Chapter 16

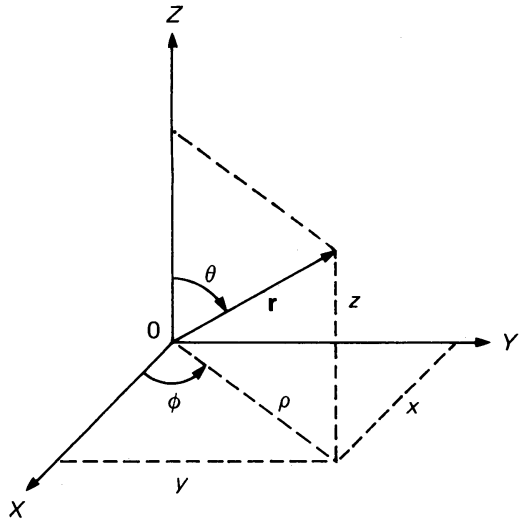
Mathematics

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Coordinate transformations

Components of a vector, **r**, in Cartesian (x, y, z), spherical (r, θ, ϕ), and cylindrical (ρ, ϕ, z) coordinates.



The following relationships exist between the Cartesian, Spherical, and Cylindrical systems:

$z = r\cos\theta$	$= z$	
$r = (x^2 + y^2 + z^2)^{1/2}$	$= (\rho^2 + z^2)^{1/2}$	
$\theta = \arccos\{z/(x^2 + y^2 + z^2)^{1/2}\}$	$= \arctan(\rho/z)$	$0 \leqslant \theta \leqslant \pi$
$\phi = \arctan(y/x)$	$= \phi$	$0 \leqslant \phi < 2\pi$
$\rho = (x^2 + y^2)^{1/2}$	$= r\sin\theta$	
$x = r\sin\theta\cos\phi$	$= \rho\cos\phi$	
$y = r\sin\theta\sin\phi$	$= \rho\sin\phi$	

Vector analysis

Vectors **a** and **b** defined in terms of the unit vectors **i**, **j**, **k** having the directions of the positive x, y, and z axes respectively:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

The scalar product:

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z$$

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The vector product:

$$\mathbf{a}\times\mathbf{b}=(a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}$$
$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}.$$

Vector identities:

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$$
$$=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b},$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\mathbf{c},$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}),$$
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\mathbf{c}-((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\mathbf{d}.$$

The vector operator ∇:

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}=\text{grad}$$

The divergence of a vector function $\mathbf{F}$:

$$\text{div}\,\mathbf{F}=\nabla\cdot\mathbf{F}=\mathbf{i}\cdot\frac{\partial\mathbf{F}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{F}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{F}}{\partial z},$$

$$\nabla\cdot\mathbf{F}=\frac{\partial\mathbf{F}_x}{\partial x}+\frac{\partial\mathbf{F}_y}{\partial y}+\frac{\partial\mathbf{F}_z}{\partial z}.$$

The curl of a vector function $\mathbf{F}$:

$$\text{curl}\,\mathbf{F}=\nabla\times\mathbf{F}=\mathbf{i}\times\frac{\partial\mathbf{F}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{F}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{F}}{\partial z}$$

$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\F_x&F_y&F_z\end{vmatrix}.$$

Differentiation formulae:

$$\nabla\cdot\phi\mathbf{u}=\phi\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\phi,$$
$$\nabla\times\phi\mathbf{u}=\phi\nabla\times\mathbf{u}+\nabla\phi\times\mathbf{u},$$
$$\nabla\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot\nabla\times\mathbf{u}-\mathbf{u}\cdot\nabla\times\mathbf{v},$$
$$\nabla\times(\mathbf{u}\times\mathbf{v})=(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}+\mathbf{u}(\nabla\cdot\mathbf{v})-\mathbf{v}(\nabla\cdot\mathbf{u}),$$
$$\nabla(\mathbf{u}\cdot\mathbf{v})=(\mathbf{u}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{u}+\mathbf{u}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{u}),$$
$$\nabla\times(\nabla\phi)=\text{curl grad}\,\phi=0,$$
$$\nabla\cdot(\nabla\times\mathbf{u})=\text{div curl}\,\mathbf{u}=0,$$

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$$\nabla \times (\nabla \times \mathbf{u}) = \text{curl curl}\,\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \cdot \nabla \mathbf{u}$$
$$= \text{grad div}\,\mathbf{u} - \nabla^2 \mathbf{u}$$
$$\nabla \cdot (\nabla \phi_1 \times \nabla \phi_2) = 0$$

In these formulas $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors and ϕ, ϕ_1, and ϕ_2 are arbitrary scalars for which the indicated derivatives exist.

Surface integrals

Divergence theorem:

$$\iiint_v \nabla \cdot \mathbf{V}\, dV = \oiint_s \mathbf{V} \cdot \mathbf{n}\, dS$$

Stokes' theorem:

$$\iint_s \mathbf{n} \cdot (\nabla \times \mathbf{V})\, dS = \oint_C \mathbf{V} \cdot d\mathbf{r}$$

Green's theorem:

$$\iiint_v [\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1]\, dV = \iint_s \mathbf{n} \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1)\, dS,$$

where $\mathbf{n}$ is the surface outward unit normal.

Differential elements

Cylindrical coordinates

Line element:

$$ds = \sqrt{(dr^2 + r^2\, d\theta + dz^2)}$$

Area elements:

$$dS_r = r\, d\theta\, dz, \quad dS_\theta = dr\, dz, \quad dS_z = r\, d\theta\, dr$$

Volume element:

$$dV = r\, d\theta\, dr\, dz$$

Spherical coordinates

Line element:

$$ds = \sqrt{(dr^2 + r^2\, d\theta + r^2 \sin^2\theta\, d\phi^2)}$$

Area elements:

$$dS_r = r^2 \sin\theta\, d\theta\, d\phi, \quad dS_\theta = r \sin\theta\, dr\, d\phi, \quad dS_\phi = r\, dr\, d\theta$$

Volume element:

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

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Definite integrals

$$\int_0^\infty t^{2n+1} e^{-at^2}\,dt = \frac{n!}{2a^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

$$\int_0^\infty t^{2n} e^{-at^2}\,dt = \frac{1\cdot 3 \ldots (2n-1)}{2^{n+1}a^n}\sqrt{\left(\frac{\pi}{a}\right)} \quad (n = 0, 1, 2, \ldots)$$

$$\int_0^\infty t^n e^{-at}\,dt = \frac{n!}{a^{n+1}} \quad (n = 1, 2, \ldots)$$

$$\int_{-\infty}^\infty e^{-\pi x^2}\,dx = 1 \qquad \int_{-\infty}^\infty e^{-Ax^2}\,dx = \left(\frac{\pi}{A}\right)^{1/2}$$

$$\int_0^\infty x e^{-x^2}\,dx = 1/2 \qquad \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt, \quad \text{error function}$$

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt, \quad \text{Fresnel integrals}$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt, \quad \text{exponential integral}$$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad \text{gamma function}$$

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt, \quad \text{Laplace transform}$$

The Fourier transform

The Fourier transform of $f(x)$ is:

$$F(s) = \int_{-\infty}^\infty f(x)e^{-i2\pi xs}\,dx$$

and

$$f(x) = \int_{-\infty}^\infty F(s)e^{i2\pi xs}\,ds.$$

There are two other equivalent versions:

$$F(s) = \int_{-\infty}^\infty f(x)e^{-ixs}\,dx,$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(s)e^{ixs}\,ds$$

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and

$$F(s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{-ixs}\, dx,$$

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(s) e^{ixs}\, ds,$$

which are also used.

Useful definitions

The *convolution* of two functions $f(x)$ and $g(x)$ is:

$$\int_{-\infty}^{\infty} f(u)g(x-u)\, du = f * g = \int_{-\infty}^{\infty} g(u)f(x-u)\, du = g * f.$$

The *autocovariance* function of $f(x)$ is:

$$f(x) \otimes f(x) = \int_{-\infty}^{\infty} f^*(u)f(u+x)\, du$$

The *autocorrelation* function of $f(x)$ is:

$$\gamma(x) = \frac{\int_{-\infty}^{\infty} f^*(u)f(u+x)\, du}{\int_{-\infty}^{\infty} f(u)f^*(u)\, du},$$

$$\gamma(0) = 1.$$

The *cross-correlation* of two functions $g(x)$ and $h(x)$ is:

$$f(x) = \int_{-\infty}^{\infty} g(u-x)h(u)\, du$$

The *power spectrum* of a function $f(x)$ is:

$$\left| \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs}\, dx \right|^2 = |F(s)|^2$$

The normalized *power spectrum* is the *power spectral density function*:

$$|\phi(s)|^2 = \frac{|F(s)|^2}{\int_{-\infty}^{\infty} |F(s)|^2\, ds}$$

The *equivalent width* of a function $f(x)$ is:

$$W_f = \frac{\int_{-\infty}^{\infty} f(x)\, dx}{f(0)}$$

The *filtering* or *interpolation* function, *sinc x*:

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

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Fourier transform theorems

Theorem	$f(x)$	$F(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}2\pi xs}\,\mathrm{d}x$
Similarity	$f(ax)$	$\frac{1}{\lvert a\rvert}F(s/a)$
Addition	$f(x)+g(x)$	$F(s)+G(s)$
Shift	$f(x-a)$	$\mathrm{e}^{-\mathrm{i}2\pi as}F(s)$
Convolution	$f(x)*g(x)$	$F(s)G(s)$
Autocovariance	$f(x)\otimes f(x)$	$\lvert F(s)\rvert^2$
Derivative	$f'(x)$	$\mathrm{i}2\pi sF(s)$

Derivative of convolution:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)*g(x)] = f'(x)*g(x) = f(x)*g'(x)$$

Parseval:

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |F(s)|^2\,\mathrm{d}s$$

Multiplication:

$$\int_{-\infty}^{\infty} f^*(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} F^*(s)G(s)\,\mathrm{d}s$$

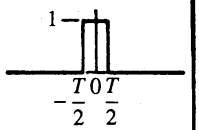
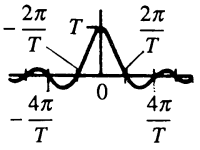
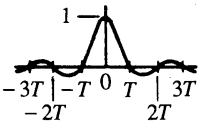
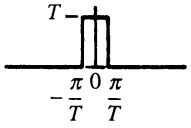
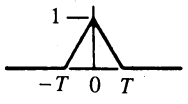
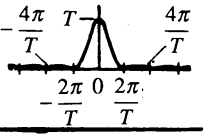
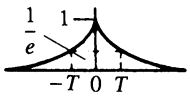
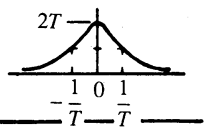
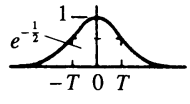
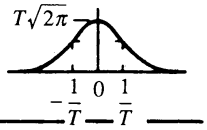
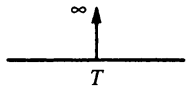
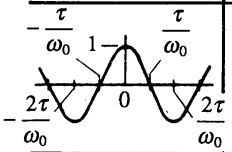
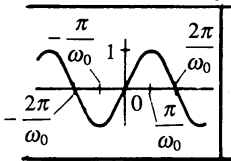
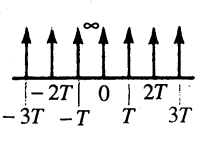
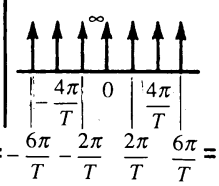
Sampling theorem:

A function whose Fourier transform is zero for $|s| > s_c$ is fully specified by values spaced at equal intervals not exceeding $\frac{1}{2}s_c^{-1}$ save for any harmonic term with zeros at the sampling points.

(Adapted from Bracewell, R. N., *The Fourier Transform and its Applications*, McGraw-Hill Book Company, 1978.)

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Fourier transform pairs

	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{-i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$	
	$\text{rect}\,\frac{t}{T} = \begin{cases} 1 & (\lvert t\rvert < T/2) \\ 0 & (\lvert t\rvert > T/2) \end{cases}$	$T\,\text{sinc}\,\frac{\omega T}{2\pi} \equiv T\,\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\text{sinc}\,\frac{t}{T} \equiv \frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T\,\text{rect}\,\frac{\omega T}{2\pi} = \begin{cases} 0 & (\lvert\omega\rvert < \frac{\pi}{T}) \\ T & (\lvert\omega\rvert > \frac{\pi}{T}) \end{cases}$	
	$\begin{cases} 1 - \frac{\lvert t\rvert}{T} & (\lvert t\rvert < T) \\ 0 & (\lvert t\rvert \geq T) \end{cases}$	$T\,\text{sinc}^2\,\frac{\omega T}{2\pi} \equiv T\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2$	
	$e^{-\frac{\lvert t\rvert}{T}}$	$\frac{2T}{(\omega T)^2 + 1}$	
	$e^{-\frac{1}{2}(\frac{t}{T})^2}$	$\sqrt{2\pi}\,T e^{-\frac{1}{2}(\omega T)^2}$	
	$\delta(t-T)$	$e^{-i\omega T}$	(Complex)
	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
	$\sin\omega_0 t$	$\frac{\pi}{i}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	(Imaginary)
	$\sum_{k=-\infty}^{\infty} \delta(t-kT) \equiv \frac{1}{T}\sum_{j=-\infty}^{\infty} e^{2\pi i j \frac{t}{T}}$	$\frac{2\pi}{T}\sum_{j=-\infty}^{\infty} \delta(\omega-\frac{2\pi j}{T}) \equiv \sum_{k=-\infty}^{\infty} e^{ik\omega T}\, T$	

(From Korn, G. A., *Basic Tables in Electrical Engineering*, McGraw-Hill, New York, 1965, with permission.)

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Spherical harmonics

$$Y_l^m(\theta,\phi)=\sqrt{\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$$

$$P_l^m(\cos\theta)=(-1)^m\sin^m\theta\left[\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right)^m P_l(\cos\theta)\right]\quad(m\leqslant l)$$

$$P_l(\cos\theta)=\frac{1}{2^l l!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right)^l(-\sin^2\theta)^l\right]$$

$$Y_l^{-m}(\theta,\phi)=(-1)^m[Y_l^m(\theta,\phi)]^*$$

$$P_l^{-m}(\cos\theta)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)$$

$$l=0\qquad Y_0^0=\frac{1}{\sqrt{(4\pi)}}$$

$$l=1\qquad Y_1^0=\sqrt{\left(\frac{3}{4\pi}\right)}\cos\theta$$

$$Y_1^1=-\sqrt{\left(\frac{3}{8\pi}\right)}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}$$

$$l=2\qquad Y_2^0=\sqrt{\left(\frac{5}{16\pi}\right)}(3\cos^2\theta-1)$$

$$Y_2^1=-\sqrt{\left(\frac{15}{8\pi}\right)}\sin\theta\cos\mathrm{e}^{\mathrm{i}\phi}$$

$$Y_2^2=\sqrt{\left(\frac{15}{32\pi}\right)}\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}$$

$$l=3\qquad Y_3^0=\sqrt{\left(\frac{7}{16\pi}\right)}(5\cos^3\theta-3\cos\theta)$$

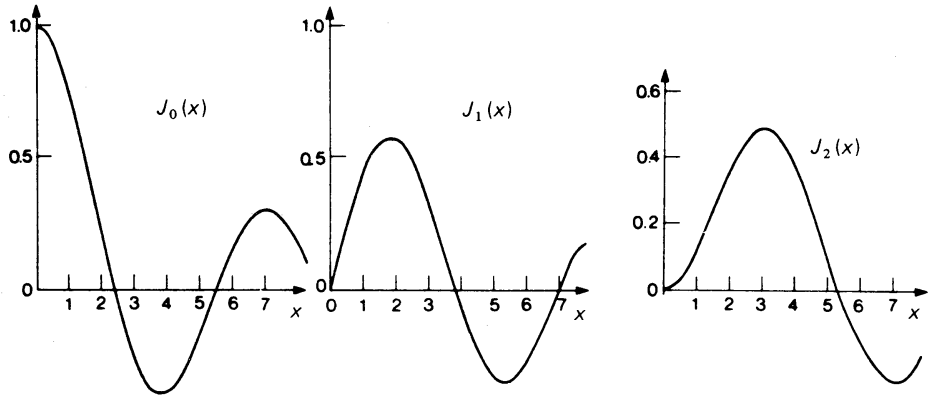
$$Y_3^1=-\sqrt{\left(\frac{21}{64\pi}\right)}\sin\theta(5\cos^2\theta-1)\mathrm{e}^{\mathrm{i}\phi}$$

$$Y_3^2=\sqrt{\left(\frac{105}{32\pi}\right)}\sin^2\theta\cos\theta\,\mathrm{e}^{2\mathrm{i}\phi}$$

$$Y_3^3=-\sqrt{\left(\frac{35}{64\pi}\right)}\sin^2\theta\,\mathrm{e}^{3\mathrm{i}\phi}.$$

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Bessel functions



$$J_n(x) = \sum_{s=0}^{s=\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} \qquad J_{n-1}(x) + J_{n+1}(x) = (2n/x)\,J_n(x)$$

$$J_0(x) = (2/\pi x)^{\frac{1}{2}} \cos(x - \pi/4)$$

(Adapted from Chantry, G. U. J., *Long-wave Optics*, Academic Press, New York, 1984.)

Numerical analysis

Taylor series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + R_n$$

$$R_n = f^{(n+1)}(\zeta)\ \frac{(x-x_0)^{n+1}}{(n+1)!} \quad (x_0 < \zeta < x)$$

Quadrature

Trapezoidal rule

$$\int_a^b f(x)\,\mathrm{d}x = h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right) - R_n$$

$R_n = \frac{1}{12}(b-a)h^2 f''(x_1) \quad (a < x_1 < b)$

$h = (b-a)/n, \quad y_k = f(a+kh), \quad k = 0, 1, \ldots, n.$

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Simpson's rule (n even)

$$\int_a^b f(x)\,\mathrm{d}x = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\right] - R_n$$

$$R_n = \tfrac{1}{90}(b-a)h^4 f^{(4)}(x_1) \quad (a < x_1 < b), \quad h = (b-a)/n.$$

Six-point Gauss–Legendre

$$\int_a^b f(x)\,\mathrm{d}x \simeq \frac{b-a}{2}\sum_{i=1}^{6} w_i f\left(\frac{z_i(b-a)+b+a}{2}\right)$$

$$\int_a^\infty f(x)\,\mathrm{d}x \simeq 2\sum_{i=1}^{6} \frac{w_i}{(1+z_i)^2} f\left(\frac{2}{1+z_i} + a - 1\right)$$

where

$$\begin{aligned} z_1 &= -z_2 = 0.238\,619\,186\,1 \\ z_3 &= -z_4 = 0.661\,209\,386\,5 \\ z_5 &= -z_6 = 0.932\,469\,514\,2 \\ w_1 &= w_2 = 0.467\,913\,934\,6 \\ w_3 &= w_4 = 0.360\,761\,573 \\ w_5 &= w_6 = 0.171\,324\,492\,4. \end{aligned}$$

Linear interpolation

$$y = \frac{(x_{k+1} - x)y_k + (x - x_k)y_{k+1}}{x_{k+1} - x_k} \quad \text{for } x_k < x < x_{k+1}.$$

Approximations

$f(x)$	Approximation	Parameters	Maximum absolute error
$\log_{10} x$	$a_1 t + a_3 t^3$	$a_1 = 0.863\,04$	6×10^{-4}
$(1/\sqrt{(10)} \leqslant x \leqslant \sqrt{(10)})$	$t = (x-1)/(x+1)$	$a_3 = 0.364\,15$	
$\arctan x$	$\dfrac{x}{1 + 0.28x^2}$		5×10^{-3}
$(-1 \leqslant x \leqslant 1)$			
$\operatorname{erf} x$	$1 - (a_1 t + a_2 t^2 + a_3 t^3)\mathrm{e}^{-x^2}$	$p = 0.470\,47$	2.5×10^{-5}
$(0 \leqslant x < \infty)$	$t = 1/(1+px)$	$a_1 = 0.348\,024\,2$	
		$a_2 = -0.095\,879\,8$	
		$a_3 = 0.747\,855\,6$	

(Adapted from Hastings, C., *Approximations for Digital Computers*, Princeton, New Jersey, 1955.)

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Curve fitting (linear least-squares)

$$y(x) = \sum_{j=0}^{n} a_j \phi_j(x), \quad \text{approximating function}$$

$$\chi^2 = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} [y_i - y(x_i)]^2,$$

where

σ_i = standard deviation of the ith observation y_i,
N = number of observations.

Minimizing χ^2, $\dfrac{\partial}{\partial a_k} \chi^2 = 0$, yields the following set of normal equations:

$$\sum_{i=1}^{N} \left[\frac{1}{\sigma_i^2} y_i \phi_k(x_i) \right] = \sum_{j=0}^{n} \left(a_j \sum_{i=1}^{N} \left[\frac{1}{\sigma_i^2} \phi_j(x_i) \phi_k(x_i) \right] \right), \text{ for } k = 0 \text{ to } n.$$

(For the case of a least-squares fit to a straight line, see Chapter 17.)

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