Chapter 10

Relativity

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Special relativity

Fundamental kinematical relations for a particle of rest mass m_0 and velocity v:

$p = m_0 v/(1 - v^2/c^2)^{1/2}$	momentum
$E = m_0 c^2/(1 - v^2/c^2)^{1/2}$	total energy
$T = E - m_0 c^2$	kinetic energy
$m = m_0/(1 - v^2/c^2)^{1/2}$	relativistic mass
$E_0 = m_0 c^2$	rest energy

From the above, the following relations can be derived:

$$E = mc^2 = (m_0^2 c^4 + c^2 p^2)^{1/2}$$

$$p = [(E/c)^2 - m_0^2 c^2]^{1/2}$$

$$v = c^2 p/E = c[1 - (m_0 c^2/E)^2]^{1/2} = p/[m_0 c^2 + (p/c)^2]^{1/2}$$

$$m = E/c^2 = [m_0^2 + (p/c)^2]^{1/2}$$

Relativistic Doppler effect:

$$1 + z = \frac{1 + (v/c)\cos\theta}{(1 - v^2/c^2)^{1/2}},$$

where

θ = angle between direction of observation and direction of motion,
$\theta = 0$ for motion directly away from observer,
$z = (\lambda_{\text{obs}} - \lambda)/\lambda$.

$$z \approx (v/c)\cos\theta \quad \text{for } v << c.$$

Lorentz transformation (Gaussian units)

4-Vector transformation

$$B'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} B_\nu \equiv a_{\mu\nu} B_\nu .$$

For a Lorentz transformation from system k to a system k' moving with a velocity parallel to the z-axis, the transformation coefficients are given by:

$$(a_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}, \quad \beta = v/c,\ \gamma = (1 - v^2/c^2)^{-1/2}.$$

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Examples of 4-vectors

$$x_\mu = (\mathbf{x}, ict),$$

where $\mathbf{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$.

$$A_\mu = (\mathbf{A}, i\phi),$$

where $\mathbf{A}$ and ϕ are the electromagnetic vector and scalar potential.

$$J_\mu = (\mathbf{J}, ic\rho),$$

where $\mathbf{J}$ and ρ are the current density and charge density.

$$k_\mu = (\mathbf{k}, i\omega/c),$$

where $\mathbf{k}$ and ω are the wave vector and frequency of a plane electromagnetic wave.

$$p_\mu = (\mathbf{p}, iE/c),$$

where $\mathbf{p}$ and E are the momentum and energy of a particle.

2nd rank tensor transformation

$$S_{\mu\nu} = \sum_{\lambda,\sigma=1}^{4} a_{\mu\lambda} a_{\nu\sigma} S_{\lambda\sigma} \equiv a_{\mu\lambda} a_{\nu\sigma} S_{\lambda\sigma}.$$

Electromagnetic field strength tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}.$$

Covariant formulation of Maxwell's equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu; \quad \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0,$$

where λ, μ, and ν are any three of the integers 1, 2, 3, 4.

Lorentz force

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}(\mathbf{J} \times \mathbf{B}).$$

$$f_\mu = \frac{1}{c} F_{\mu\nu} J_\nu = \frac{1}{4\pi} F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda} = \frac{\partial T_{\mu\nu}}{\partial x_\nu} = \left(\mathbf{f}, \frac{i}{c}\mathbf{E}\cdot\mathbf{J}\right),$$

where $T_{\mu\nu}$ is the electromagnetic stress–energy–momentum tensor:

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F_{\lambda\nu} + \tfrac{1}{4}\delta_{\mu\nu}F_{\lambda\sigma}F_{\lambda\sigma}\right].$$

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Robertson–Walker line element (homogeneous and isotropic universe)

$$ds^2 = c^2\,dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right],$$

where

$R(t)$ = radius of curvature of the universe,

r, θ, ϕ = co-moving spherical coordinates (a co-moving observer is an observer at rest with respect to matter in his vicinity),

k = curvature index $= 0, \pm 1$ ($k = 1$, elliptical closed space; $k = 0$, Euclidean flat space; $k = -1$, hyperbolic open space).

Einstein field equations

$$\frac{3\dot{R}^2}{R^2} + \frac{3kc^2}{R^2} = 8\pi G\rho + \Lambda c^2,$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = -\frac{8\pi GP}{c^2} + \Lambda c^2,$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3}\left[\rho + \frac{3P}{c^2}\right],$$

where

ρ = mean density of matter and energy,

Λ = cosmological constant,

P = hydrodynamic pressure of matter and radiation,

G = gravitational constant,

$H_0 \equiv \dot{R}_0/R_0$, Hubble constant,

$q_0 \equiv -\ddot{R}_0/R_0H_0^2$, deceleration constant, where the subscript zero denotes the present value.

Friedmann universes ($\Lambda = 0$)

$$\rho_0 + \frac{3P_0}{c^2} = \frac{3H_0q_0}{4\pi G},$$

$$\frac{kc^2}{R_0^2} = \frac{4\pi G}{3q_0}\left[\rho_0(2q_0 - 1) - \frac{3P_0}{c^2}\right].$$

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For epochs after the decoupling of matter, P_0 is approximately 0, and we have

Curvature	Space	q_0	Ω_0[a]	Density ρ_0	Expansion
$k=+1$	Closed	$>1/2$	>1	$>\dfrac{3H_0^2}{8\pi G}$	Turns eventually into contraction
$k=0$	Flat (Euclidean)	$1/2$	1	$\rho_c=\dfrac{3H_0^2}{8\pi G}$ [b]	Stops in infinite future
$k=-1$	Open	$0\leqslant q_0<1/2$	$0\leqslant\Omega_0<1$	$<\dfrac{3H_0^2}{8\pi G}$	Forever

[a] The 'density parameter' $\Omega\equiv\rho/\rho_c$, where ρ_c is the critical closure density (i.e. for the case $q_0=1/2$).

[b] With $H_0=50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the present critical density becomes $\rho_c=4.7\times10^{-30}\,\mathrm{g\,cm^{-3}}$.

Useful relationships and quantities (Friedmann universe)

Differential volume:

$$\mathrm{d}V=\frac{R_H^3}{(1+z)^3}\,\frac{\{q_0z+(q_0-1)[(1+2q_0z)^{1/2}-1]\}^2}{q_0^4(1+2q_0z)^{1/2}}\,\mathrm{d}\Omega\,\mathrm{d}z,$$

where

$$R_H=\frac{c}{H_0},\quad\text{the Hubble radius.}$$

Time differential:

$$\mathrm{d}t=-\mathrm{d}z/[(1+z)^2H_0(1+2q_0z)^{1/2}].$$

Look-back time:

$$\tau=-\int_0^z\mathrm{d}t,$$

$q_0=0$,

$$\tau=\frac{1}{H_0}(1-1/(1+z));\quad\lim_{z\to\infty}\tau=1/H_0,$$

$q_0=1/2$,

$$\tau=\frac{2}{3H_0}(1-1/(1+z)^{3/2});\quad\lim_{z\to\infty}\tau=2/3(1/H_0),$$

$q_0=1$,

$$\tau=\frac{1}{H_0}[(2z+1)^{1/2}/(z+1)+2\tan^{-1}(2z+1)^{1/2}-1-\pi/2];$$

$$\lim_{z\to\infty}\tau=0.57(1/H_0),$$

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where

H_0 = Hubble constant,
q_0 = deceleration constant,
z = redshift.

Redshift-magnitude relationship (Friedmann universe):

$$m_{\text{bol}} = 5\log\left\{\frac{c}{H_0 q_0^2}[1 - q_0 + q_0 z + (q_0 - 1)(2q_0 z + 1)^{1/2}]\right\} + M_{\text{bol}} + 25 = 5\log D_{\text{L}} + M_{\text{bol}} + 25.$$

where m_{bol} is the apparent bolometric magnitude of a source with absolute bolometric magnitude M_{bol} and redshift z.
Expanded in powers of z:

$$m_{\text{bol}} = 5\log\left(\frac{zc}{H_0}\right) + 1.086(1 - q_0)z + \cdots + M_{\text{bol}} + 25,$$

where H_0 is in $\text{km s}^{-1}\,\text{Mpc}^{-1}$; zc is in km s^{-1}, and z is the observed redshift.

$$D_{\text{L}} = \frac{c}{H_0 q_0^2}[1 - q_0 + q_0 z + (q_0 - 1)(2q_0 z + 1)^{1/2}]$$

$$\approx \frac{cz}{H_0}[1 + 0.5z(1 - q_0)], \quad q_0 z << 1.$$

$m_{\text{bol}} - M_{\text{bol}} = m - M - K - A$, $m - M$ = observed distance modulus, for heterochromatic magnitudes, where K = redshift correction, A = interstellar absorption.

$$K = 2.5\log(1 + z) + 2.5\log\frac{\int_0^\infty I(\lambda)S(\lambda)\,\mathrm{d}\lambda}{\int_0^\infty I\left(\frac{\lambda}{1+z}\right)S(\lambda)\,\mathrm{d}\lambda}\,[\text{mag}],$$

where $I(\lambda)$ is the incident energy flux per unit wavelength and $S(\lambda)$ is the photometer response function.

Angular diameter–redshift relationship (Friedmann universe)

$$\theta = \frac{l(1 + z)^2}{D_{\text{L}}},$$

where

θ = apparent angular diameter of source,
l = linear diameter of spherical source,
D_{L} = luminosity distance.

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Observed energy flux density (Friedmann universe)

$$S_0(E_0) = \frac{P_e((1+z)E_0)}{4\pi D_L^2},$$

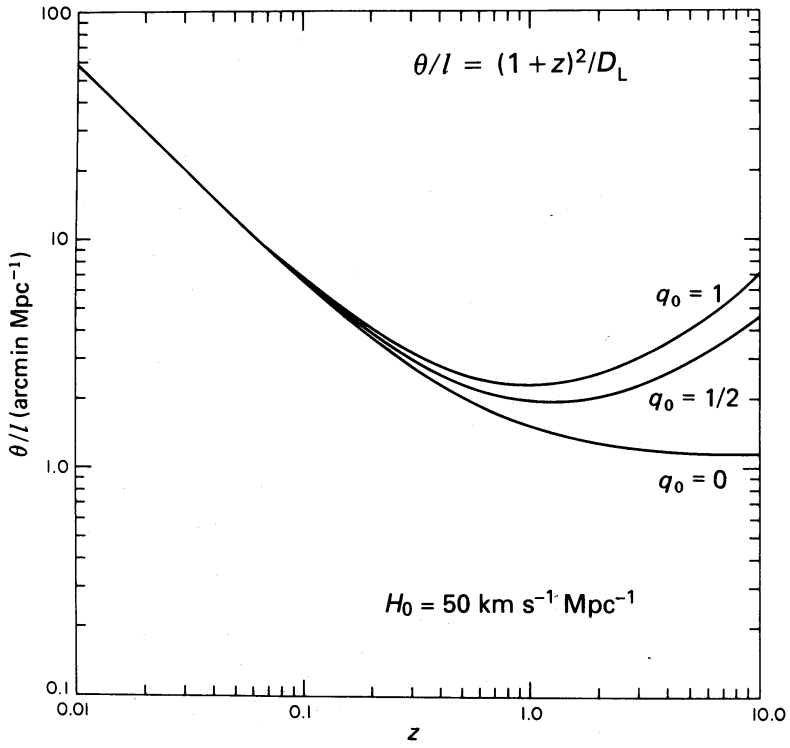
where P_e is the monochromatic power emitted at the energy $(1+z)E_0$, and S_0 is the observed monochromatic energy flux density at the energy E_0. The observed energy flux density integrated from E_1 to E_2 is:

$$F_0(E_1, E_2) = \frac{1}{4\pi D_L^2} \int_{(1+z)E_1}^{(1+z)E_2} p_e(E)\, dE,$$

where $p_e(E)$ is the differential power emitted per unit energy in the emitted rest frame.

Redshift functions

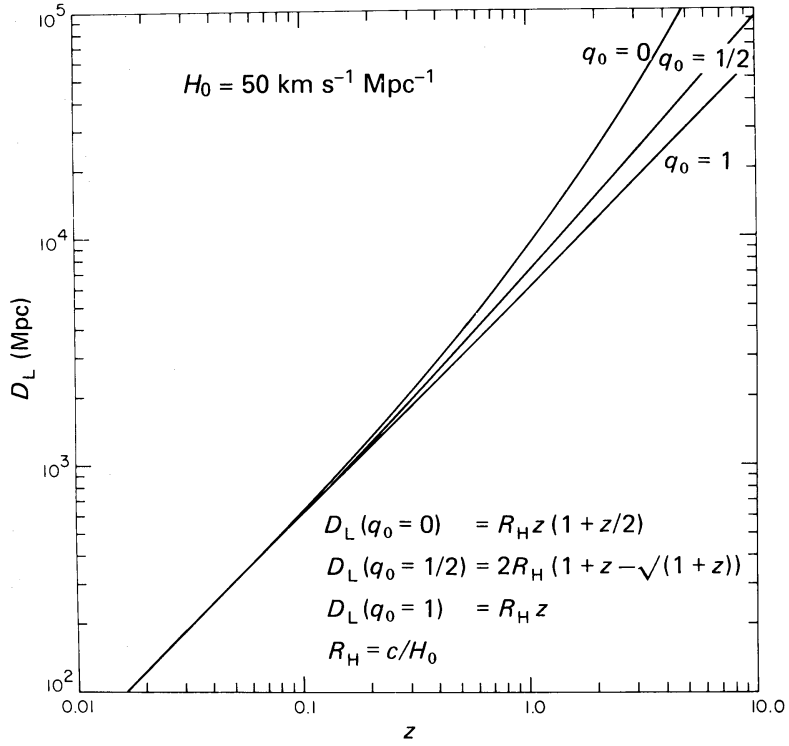
Angular size vs. redshift



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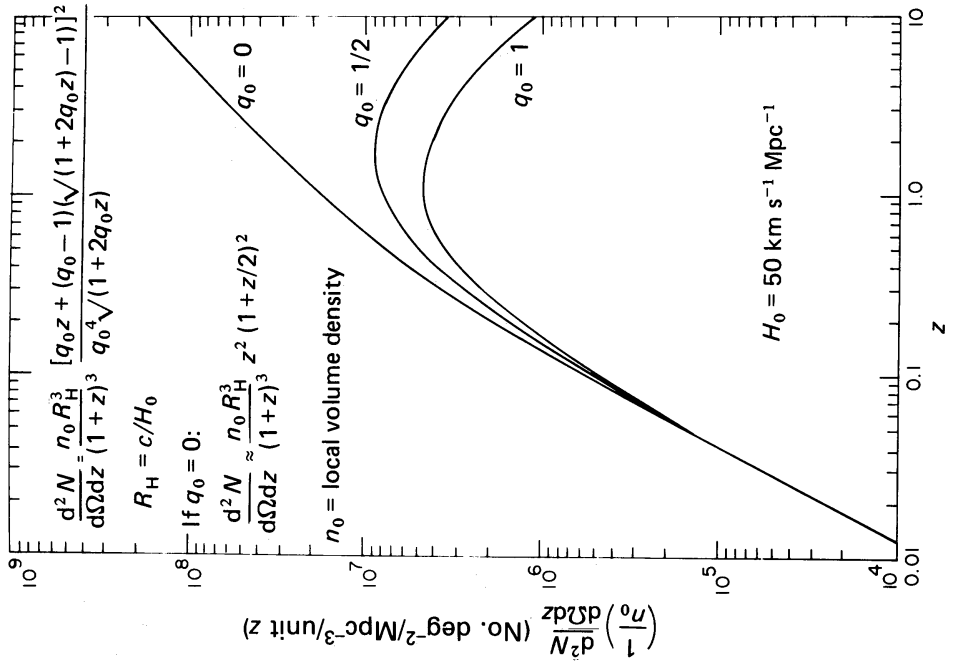
Redshift functions (cont.)

Luminosity distance vs. redshift

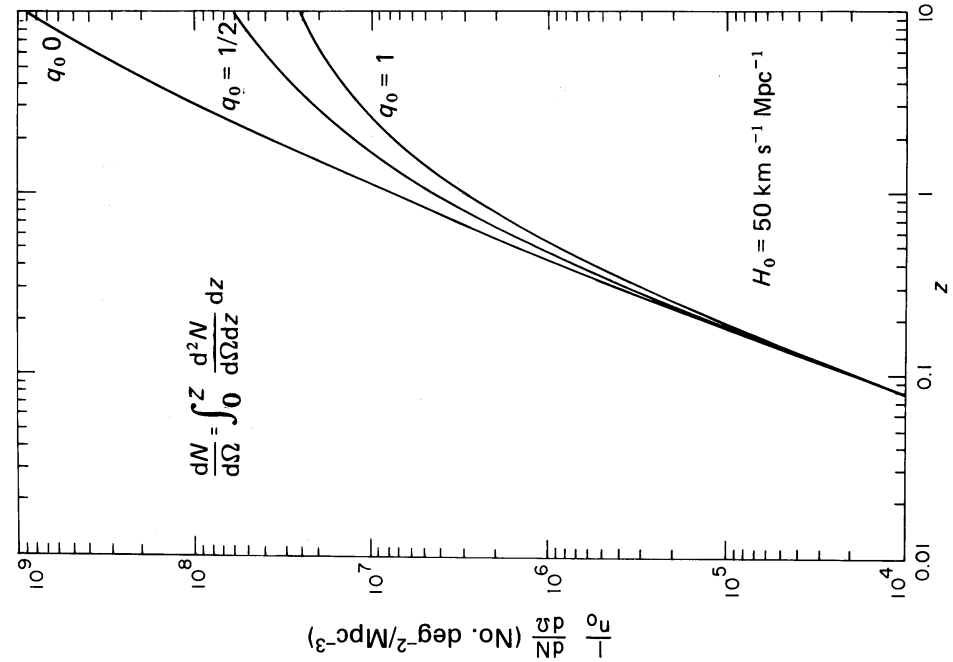


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Angular density of sources per unit z vs. redshift.



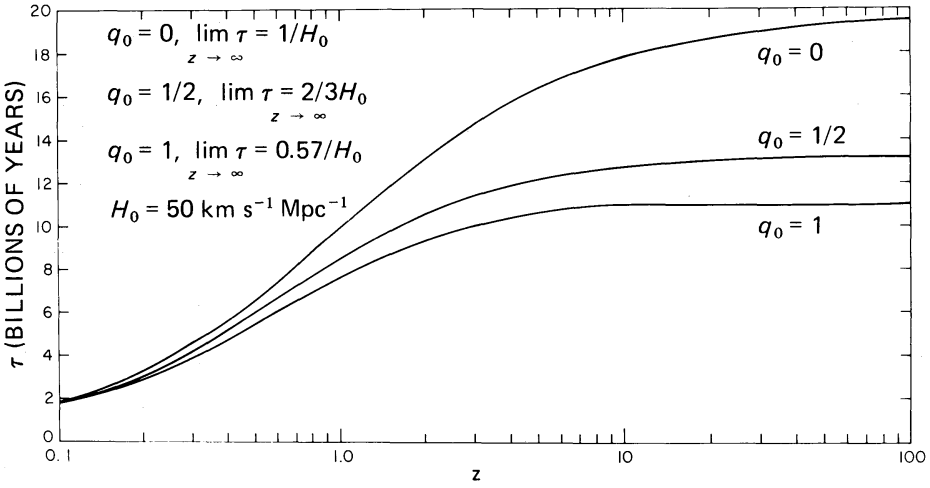
Integrated angular density vs. redshift.



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Redshift functions (cont.)

Look-back time



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